Proc. Indian Acad. Sci. (Math. Sci.) (2026) 136:11
https://doi.org/10.1007/s12044-026-00860-x

On the reducibility of a certain type of rank 3 uniform oriented matroid by a point

C P ANIL KUMAR*

Harish-Chandra Research Institute, (Department of Atomic Energy, Government of India), Chhatnag Road, Jhunsi, Prayagraj (Allahabad), Uttar Pradesh 211019, India.
e-mail: akcp1728@gmail.com

MS received 13 August 2023; revised 14 November 2025; accepted 18 December 2025
© The Author(s), under exclusive licence to Indian Academy of Sciences 2026

Abstract. For a positive integer $n \geq 3$, the sides and diagonals of a convex n-gon divide the interior of the convex n-gon into finitely (polynomial in n) many regions bounded by them. In this article, we associate to every region a unique n-cycle in the symmetric group S_n of a certain type (defined as 2-standard consecutive cycle) by studying point arrangements in the plane. Then we find that there are more (exponential in n) number of such cycles leading to the conclusion that not every region labelled by a cycle appears in every convex n-gon. In fact most of them do not occur in any given single convex n-gon. Later in the main theorem of this article we characterize combinatorially those cycles (defined as definite cycles) whose corresponding regions occur in every convex n-gon and those cycles (defined as indefinite cycles) whose corresponding regions do not occur in every convex n-gon. As a consequence we characterize those one point extensions of a uniform rank 3 convex oriented matroid for which the one point extension is reducible by the, one point, when it lies inside the convex hull.

Keywords. Point arrangements in the plane; Line cycles; Two standard consecutive structure; Definite and indefinite regions; Matroids; Reducibility of a matroid by a point.

Mathematics Subject Classification. Primary: 52C40, Secondary: 52C35, 05B25, 51D20.

1. Introduction

The regions of a convex n-gon when divided by the diagonals has been studied in various contexts in the literature ([1,2] [Chapter 9], [3] [Chapter 3], the OEIS sequence A006522 [4]). The problems on points in general position are treated in Chapter 8 of Brass et al. [5]. Here, in [5], two convex 7-gons with different diagonal arrangements is given. The vertices of a convex n-gon can be thought of as a point arrangement in the plane $\{z = 1\} \subset \mathbb{R}^3$ and hence can be thought of as a vector configuration in $\mathbb{R}^3$. In general, point arrangements/vector configurations V in $\mathbb{R}^n$ can be associated with various types of combinatorial structures such as (refer [6])

(1) the chirotope of V upto a sign,
(2) the set of covectors of V,
(3) the collection of cocircuits of V,

Published online: 30 May 2026

(4) the set of vectors of V,
(5) the set of circuits of V.

Each combinatorial structure can be axiomatized to an axiomatic system and can be defined abstractly as an *oriented matroid* and all these five axiomatic systems define equivalent notions of what an abstract oriented matroid is. Although the axiom systems of oriented matroids describe the data arising from vector configurations very well, it is not true that every oriented matroid corresponds to a real vector configuration. In other words, there are oriented matroids that are not realizable.

If a vector configuration $V = \{v_i \mid 1 \leq i \leq n\} \subset \mathbb{R}^n$ has no positive linear dependences, that is, $\sum_i \lambda_i v_i = 0$ with $\lambda_i \geq 0$ then this constitutes what is abstractly known as an acyclic oriented matroid, that is, an oriented matroid, that does not contain any positive circuits. In this scenario, there exists a linear functional $l : \mathbb{R}^n \to \mathbb{R}$ such that $l(v_i) > 0$ for all $1 \leq i \leq n$.

For construction and analysis point of view of oriented matroids, it is of interest to study and describe the set of all single point extensions of a given oriented matroid. Single point extensions of general matroids (that is, matroids without an orientation structure) have been studied in a lot more detail by Crapo [7]. One point extensions of oriented matroids are described in Chapter 7 of [8].

In this article, we consider certain n-point arrangements in an affine plane in $\mathbb{R}^3$ or certain vector configurations in the space $\mathbb{R}^3$, that is, acyclic oriented uniform matroids of rank 3, where the n points are in a convex position in the affine plane. We also consider its one point extensions where the additional point p is contained in the interior of a region of the convex hull of the n points when the hull is divided by its diagonals. The region becomes the *residence* of the point p, (see [9], page 211 for definition of "*residence*", also see Definition 2.2). To study the residence of p, we associate, to any region of the convex n -gon when divided by diagonals, a unique 2-standard consecutive cycle. The method of associating cycle invariants as a combinatorial model to point arrangements in the plane has already been explored by Goodman and Pollack [10,11]. A similar method is explained in [12] [Chapter 10]. The classical notions of convexity theory of vector configurations such as convex hull, faces of convex polytopes have been generalized to oriented matroids in Vergnas [13].

2. Preliminaries

Now we mention a few definitions to proceed further.

DEFINITION 2.1

(Point Arrangement in the Plane) Let $\mathcal{P}_n = \{P_1, P_2, \ldots, P_n\}$ be a finite set of n points in the plane $\mathbb{R}^2$. Then $\mathcal{P}_n$ is said to be a point arrangement in the plane if no three points are collinear.

Let $P_n = \{p_1, p_2, \cdots, p_n\}$ be a point arrangement on a plane H not passing through origin in $\mathbb{R}^3$. Then the corresponding vector configuration, that is, thinking of the points of the set P_n as vectors in $\mathbb{R}^3$, it gives rise to a uniform oriented matroid M. Now the lines joining pairs of points in P_n divide the plane H into a finite number of convex polygonal

regions. Introducing a point p_{n+1} in the interior of a particular convex region gives rise to what is known as a one point extension of P and its associated vector configuration is a uniform oriented matroid $\tilde{M}$ consisting of $n+1$ points.

DEFINITION 2.2

(Definition of Residence of the Newly Added Point) The interior of the convex region in which p_{n+1} lies is called the residence of p_{n+1}. Also see L. M. Kelly and W. O. J. Moser [9], page 211 for definition of "*residence*".

DEFINITION 2.3

Let M be an oriented matroid on a set E. An extension of M is an oriented matroid $\tilde{M}$ on a ground set $\tilde{E}$ that contains E such that the restriction of $\tilde{M}$ to E is M. We say $\tilde{M}$ is a one point extension of the matroid M if $| \tilde{E}\backslash E | = 1$, that is, $\tilde{E} = E \cup \{p\}$ for some $p \notin E$. Also see Definition 7.1.1, page 281 in [8].

Let M denote a rank d oriented matroid on $E = \{1, 2, \cdots, n\}$, and let $\widetilde{\mathcal{R}(M)}$ denote the set of real $n \times d$ matrices realizing M. The realization space $\mathcal{R}(M)$ is defined as the quotient of the topological space $\widetilde{\mathcal{R}(M)} \subseteq \mathbb{R}^{nd}$ by the canonical action of the group $GL(d, \mathbb{R})$ of nonsingular $d \times d$ matrices. Deleting a point $e \in E$ in M induces a natural map $\pi_e : \mathcal{R}(M) \longrightarrow \mathcal{R}(M\backslash e)$. Each fiber of this map is a convex set, namely it is the *residence* of the point e in a realization of M.

DEFINITION 2.4

(Reducibility of an oriented matroid by a point) We say the oriented matroid M is reducible by a point $e \in E$ provided the deletion map $\pi_e : \mathcal{R}(M) \longrightarrow \mathcal{R}(M\backslash e)$ is surjective. In other words, M is reducible by e if every real realization of the minor $M\backslash e$ extends to a realization of M. Also see the definition on page 341, before Example 8.1.1 in [8] or [14].

DEFINITION 2.5

Let $n \geq 3$ be a positive integer. We say C_n is the convex uniform matroid of rank 3 if it consists of a set E_n of n points say $E_n = \{1, 2, \cdots, n\}$ and for any $1 \leq i < j < k < l \leq n$ the signed circuits of the matroid are given as $X = \{i < j < k < l\} \subseteq E_n$ where we have $X^+ = \{i, k\}$, $X^- = \{j, l\}$ or vice-versa with $X = X^+ \sqcup X^-$ as the signed circuit.

Remark 2.6. So in any realization of C_n there are n points in convex position in the affine plane $\mathbb{R}^2$. Also in a realization matrix $A = (v_1, v_2, \cdots, v_n) \in Mat_{3,n}(\mathbb{R})$ of C_n we have $\mathrm{Det}(v_i, v_j, v_k) > 0$ for every $1 \leq i < j < k \leq l$.

Remark 2.7. It can be shown that this matroid C_n is acyclic, that is, there exists a linear functional $l_A \in \mathrm{Hom}(\mathbb{R}^3, \mathbb{R})$ such that $l_A(v_i) > 0$ for all $1 \leq i \leq n$ for any such realization matrix A of C_n. For three column vectors $x, y, z \in \mathbb{R}^3$ if the column vector $x \times y$ denotes the cross product in $\mathbb{R}^3$ then we have $\mathrm{Det}(x, y, z) = x \cdot (y \times z)$. Hence

if $A = (v_1, v_2, \cdots, v_n) \in Mat_{3,n}(\mathbb{R})$ realizes C_n then we have $\mathrm{Det}(v_i, v_j, v_k) > 0$ for every $1 \leq i < j < k \leq l$. So choose $l : \mathbb{R}^3 \to \mathbb{R}$ such that for a column vector $v \in \mathbb{R}^3$,

$$l(v) = v \cdot \left(\sum_{i=1}^{n-1} v_i \times v_{i+1} + v_n \times v_1 \right).$$

Then we have $l(v_i) > 0$ for all $1 \leq i \leq n$.

DEFINITION 2.8

(The one-point extension D_n for $n \geq 3$) We say $D_n = C_n \cup \{n+1\}$ is a one point extension of the matroid C_n consisting of an additional point $p = n + 1$ if in any realization, the point p is in general position contained in the interior of the convex hull of n points of C_n. So if $B = (v_1, v_2, \cdots, v_n, v_{n+1}) \in Mat_{3,n+1}(\mathbb{R})$ is any realization of D_n then $A = (v_1, v_2, \cdots, v_n)$ is a realization of C_n and we have $\mathrm{Det}(v_i, v_{i+1}, v_{n+1}) > 0$ for any $1 \leq i \leq n-1$ and $\mathrm{Det}(v_n, v_1, v_{n+1}) > 0$.

DEFINITION 2.9

(Definite Region and Indefinite Region) We say a region/residence R of a point p in a convex n-gon when divided by diagonals is a reducible region or definite region if the corresponding matroid D_n is reducible by the point p. We say it is an indefinite region if D_n is not reducible by the point $p \in R$.

In [14] and also in the book, [8], Chapter 8, the authors introduce the notion of isolated points for uniform oriented matroids and study the concept of reducibility along with it. In fact they prove in [14] Theorem 3.4, page 397 that, in a rank 3 uniform oriented matroid M, any point $p \in M$ is either isolated or M is reducible by p. Using this they prove the main result that, if M is a uniform rank 3 oriented matroid with at most eight points then the realization space $\mathcal{R}(M)$ of M is contractible, and hence M satisfies the isotopy property (which states that the space $\mathcal{R}(M)$ is path-connected). For an oriented matroid M, an ordering $(e_1, e_2, \cdots, e_n) \in E$ is called a reduction sequence for M if either $d = n$ or M is reducible by e_n and $(e_1, e_2, \cdots, e_{n-1})$ is a reduction sequence for $M \backslash e_n$. Now the following holds:

PROPOSITION 2.10

Let M be an oriented matroid.

(1) (Lemma 2.1 [14], Lemma 8.2.1 [8]) Suppose M is reducible by e then $\mathcal{R}(M)$ is homotopy equivalent to $\mathcal{R}(M \backslash e)$.

(2) (Corollary 2.2 [14]) If M admits a reduction sequence then its realization space $\mathcal{R}(M)$ is contractible.

(3) (Corollary 3.5 [14]) If M has six or less points, then M is reducible by each of its points.

Proposition 2.10 can be applied to the matroid D_n in main Theorem **Ω** of the article giving rise to some more consequences.

The hyperplanes of a matroid have been studied by D. J. A. Welsh [15] for a general matroid and as part of generalized convexity theory for oriented matroids, it has been studied by M. L. Vergnas [13]. Now we introduce generic realization spaces.

2.1 *Generic Realization Spaces*

Let M be an oriented matroid of rank d on $E = \{1, 2, \cdots, n\}$. Let $\mathcal{H} = \{H \mid H$ is a hyperplane of the matroid M, that is, a flat of rank $d-1\}$. Then we have $\mathcal{C}^* = \{E\backslash H \mid H \in \mathcal{H}\}$ is the set of cocircuits and in an oriented matroid we have the decomposition $E\backslash H = (E\backslash H)^+ \sqcup (E\backslash H)^-$ as a signed cocircuit representing the two sides of the hyperplane H. In a realization matrix $A \in Mat_{d,n}(\mathbb{R})$ of the matroid M, let H_A denote the realization hyperplane (a subspace of $\mathbb{R}^d$ of dimension $d-1$) corresponding to a hyperplane H of the matroid M. Let $\widetilde{\mathcal{R}_{gen}(M)}$ denote the set of real $(n \times d)$-matrices $A \in Mat_{d,n}(\mathbb{R})$ realizing M for which the hyperplanes of the matroid M form a hyperplane arrangement which satisfy the following intersection property.

If $H_1, H_2, \cdots H_r$ are r distinct hyperplanes of the matroid and $H_1 \cap H_2 \cap \cdots \cap H_r$ is a flat of rank $k \geq 0$ then we must have $(H_1)_A \cap (H_2)_A \cap \cdots \cap (H_r)_A$ is a subspace in $\mathbb{R}^d$ of dimension $\max(k, d-r)$. The generic realization space $\mathcal{R}_{gen}(M)$ is defined as the quotient of the topological space $\widetilde{\mathcal{R}_{gen}(M)} \subseteq \widetilde{\mathcal{R}(M)} \subseteq \mathbb{R}^{nd}$ by the canonical action of the group $GL(d, \mathbb{R})$ of nonsingular $d \times d$ matrices.

Remark 2.11. Let C_n be the convex matroid given in Definition 2.5. Then $\mathcal{R}_{gen}(M)$ consists of those realizations of C_n where the diagonals of the convex n-gon have the property that, no three diagonals have a common point of intersection.

We formally define what a side of a convex n-gon and what a diagonal of a convex n-gon are.

DEFINITION 2.12

(Side and Diagonal of a Convex n -gon) Let $\mathcal{P}_n = \{P_1, \ldots, P_n\}$ be a point arrangements in $\mathbb{R}^2$ such that the points form a convex n -gon in the anticlockwise manner $P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P_n \longrightarrow P_1$. A side of the convex n -gon is a line P_iP_j with $j \equiv i \pm 1 \mod n$ where $1 \leq i \neq j \leq n$. A diagonal of the convex n -gon is a line P_iP_j with $j \not\equiv i \pm 1 \mod n$ where $1 \leq i \neq j \leq n$.

Now we see in the definition below when a convex n-gon has generic diagonals.

DEFINITION 2.13

(Convex n -gon with Generic Diagonals) Let $\mathcal{P}_n = \{P_1, \ldots, P_n\}$ be a point arrangement in $\mathbb{R}^2$ such that the points form a convex n -gon in the anticlockwise manner $P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P_n \longrightarrow P_1$. We say the convex n -gon has generic diagonals if for every three distinct pairs of subscripts $\{i_t, j_t\}$, $1 \leq i_t \neq j_t \leq n$, $1 \leq t \leq 3$ with

$$\{i_1, j_1\} \cap \{i_2, j_2\} \cap \{i_3, j_3\} = \emptyset,$$

$P_{i_1}P_{j_1}$, $P_{i_2}P_{j_2}$, $P_{i_3}P_{j_3}$ do not concur in the plane $\mathbb{R}^2$.

2.2 *The Number of Regions in a Convex n -gon with Generic Diagonals*

We consider the regions of a convex n -gon which has generic diagonals and compute the number of regions formed by the diagonals and sides.

Remark 2.14. (A Region) A region is defined to be a connected component of the interior of the convex n -gon when the diagonals and sides are removed.

Now we state the theorem.

Theorem 2.15. *(Chapter 9, pp. 99-107 in R. Honsberger [2], J. W. Freeman [1]) Let $\mathcal{P}_n = \{P_1, \ldots, P_n\}$ be a point arrangement in $\mathbb{R}^2$ such that the points form a convex n -gon in the anticlockwise manner $P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P_n \longrightarrow P_1$ and which has generic diagonals. Then the number of regions formed by the convex n -gon is given by*

$$\frac{(n-1)(n-2)(n^2-3n+12)}{24}.$$

3. On the 2 -Standard Consecutive Structure of an *n* -Cycle and the Regions of a Convex *n* -gon

In this section, for a positive integer i, we define i -standard consecutive structure on an n -cycle. We associate to a region in a convex n -gon formed by the diagonals, a 2 -standard consecutive cycle.

Now we introduce the structure on a permutation as follows.

DEFINITION 3.1

(i -Standard Cycle) We say an n -cycle $(1 = a_1 a_2 \ldots a_n)$ is an i -standard cycle if there exists a way to write the integers a_i, $i = 1, \ldots, n$ as i sequences of inequalities as follows:

$$\begin{aligned}
&a_{11} < a_{12} < \ldots < a_{1j_1}\\
&a_{21} < a_{22} < \ldots < a_{2j_2}\\
&a_{31} < a_{32} < \ldots < a_{3j_3}\\
&\vdots\\
&a_{i1} < a_{i2} < \ldots < a_{ij_i}
\end{aligned}$$

where $\{a_{st} \mid 1 \le s \le i, 1 \le t \le j_s\} = \{a_1, a_2, \ldots, a_n\} = \{1, 2, \ldots, n\}$, $j_1 + j_2 + \ldots + j_i = n$ and i is minimal, that is, there exists no smaller integer with such property and furthermore that $a_{s(t+1)}$ occurs to the right of a_{st} for every $1 \le s \le i$ and $1 \le t \le j_s - 1$ in this cycle arrangement $(a_1 = 1, a_2, \ldots, a_n)$. See Remark 3.3 below.

Define i -standard consecutive structure on an n -cycle as follows.

DEFINITION 3.2

(i -Standard Consecutive Cycle) We say an n -cycle $(1 = a_1 a_2 \dots a_n)$ is a consecutive i -standard cycle or a i -standard consecutive cycle if we have

$$a_{s1} < a_{s2} < \ldots < a_{sj_s}$$

and in addition $a_{st} = a_{s1} + (t-1)$, $1 \le t \le j_s$, $1 \le s \le i$ where $\{a_{st} \mid 1 \le s \le i, 1 \le t \le j_s\} = \{a_1, a_2, \dots, a_n\} = \{1, 2, \dots, n\}$, $j_1 + \ldots + j_s = n$ and $a_{s(t+1)}$ occurs to the right of a_{st} for every $s = 1, \dots, i$ and $1 \le t \le j_s - 1$ in this cycle arrangement $(1 = a_1 a_2 \dots a_n)$ and i is minimal, that is, there exists no smaller integer with such property. If the minimal value of i is two then we say that the cycle has 2 -standard consecutive structure.

Remark 3.3. For a permutation $w = w_1 w_2 \cdots w_n$ in word notation, if it can be decomposed into a minimum of k (and not k' with $k' < k$) increasing subsequences such that every element w_i appears in some unique increasing subsequence, then the value k is known to be the length of the longest decreasing subsequence of w.

Example 3.4. The 5 -cycle (14523) it is a 2 -standard consecutive cycle. However it has the following two 2 -standard structures.

- $1 < 4 < 5, 2 < 3$ (not consecutive).
- $1 < 2 < 3, 4 < 5$ (consecutive).

DEFINITION 3.5

(Line Cycle at a Point) Let $\mathcal{P}_n = \{P_1, P_2, \dots, P_n\}$ be a point arrangement in the plane. Fix $P_i \in \mathcal{P}_n$ for some $1 \le i \le n$. Consider the $(n-1)$ lines L^i_j joining P_i, P_j for $1 \le j \le n$, $j \ne i$. An anticlockwise traversal around the point P_i cuts the lines L^i_j in a cycle $\sigma_i \in S_{n-1}$, a symmetric group over the elements $\{1, 2, \dots, i-1, i+1, \dots, n\}$. This cycle is defined to be the line cycle at P_i for the arrangement $\mathcal{P}_n$.

DEFINITION 3.6

Let $(a_1 a_2 \cdots a_n)$ be a n-cycle in S_n. Then a sub-cycle is defined to be a cycle obtained by deleting some of the $a_i's$, that is, a k sub-cycle is of the form $(a_{i_1} a_{i_2} \cdots a_{i_k})$ for some $1 \le i_1 < i_2 < \cdots < i_k \le n$.

The following result enables us to identify regions by their 2-standard consecutive cycles.

Theorem 3.7. *Let $\mathcal{P}_n = \{P_1, \dots, P_n\}$ be a point arrangement in $\mathbb{R}^2$ such that the points form a convex n -gon in the anticlockwise manner $P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P_n \longrightarrow P_1$ and which has generic diagonals. Let R be a region and $P_{n+1} \in R$. Then the line cycle of P_{n+1} for the point arrangement $\mathcal{P}_n \cup \{P_{n+1}\}$ is a 2-standard consecutive cycle and the cycle is independent of any point $P_{n+1} \in R$ and depends only on the region R. Moreover different regions of the convex n -gon are associated to different cycles.*

Proof. Since the point $P_{n+1} \in R$ and does not lie on the diagonals, the finite set $\mathcal{P}_n \cup \{P_{n+1}\}$ is a point arrangement. An anticlockwise traversal around the point P_{n+1} gives a cycle $(1 = a_1 a_2 \ldots a_n)$ which has the property that it is obtained by interlacing in some manner two sequences $1 < 2 < \ldots < i$ and $i+1 < i+2 < \ldots < n$ for some $2 < i < n$ giving rise to a 2-standard consecutive cycle. It is clear that the line cycle depends only on the region R. If R and S are two different regions then there exist three subscripts $i < j < k$ such that $\Delta P_i P_j P_k$ contains R and $\Delta P_i P_j P_k$ does not contain S. In the cycle associated to R the subscripts i, j, k appear as the sub-cycle (ikj) and in the cycle associated to S the subscripts i, j, k appear as the sub-cycle (ijk) and clearly they are different. □

DEFINITION 3.8

(2-Standard Consecutive Cycle of a Region) Let $\mathcal{P}_n = \{P_1, \ldots, P_n\}$ be a point arrangement in $\mathbb{R}^2$ such that the points form a convex n -gon in the anticlockwise manner $P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P_n \longrightarrow P_1$ and which has generic diagonals. Let R be a region. The line cycle associated to any point $P \in R$ is defined to be the 2-standard consecutive cycle of the region R. Using Theorem 3.7, the cycle is well defined, unique and has the 2-standard consecutive structure.

DEFINITION 3.9

(Isomorphism of Point Arrangements) Let $\mathcal{P}_n^j = \{P_1^j, P_2^j, \ldots, P_n^j\}$, $j = 1, 2$ be two point arrangements in the plane $\mathbb{R}^2$. Then a bijection $\delta : \mathcal{P}_n^1 \longrightarrow \mathcal{P}_n^2$ is an isomorphism if for any four points $A, B, C, D \in \mathcal{P}_n^1$, D is in the interior of the triangle formed by A, B, C if and only if $\delta(D)$ is in the interior of the triangle formed by $\delta(A), \delta(B), \delta(C)$.

We say the isomorphism δ is an orientation preserving isomorphism if for any three points the $A, B, C \in \mathcal{P}_n^1$ the orientation $A \longrightarrow B \longrightarrow C \longrightarrow A$ of the triangle ΔABC and the orientation $\delta(A) \longrightarrow \delta(B) \longrightarrow \delta(C) \longrightarrow \delta(A)$ of the triangle $\Delta\delta(A)\delta(B)\delta(C)$ agree. In [5], this is defined as an equivalence relation for order types of point arrangements. In this case $\mathcal{P}_n^1$ and $\mathcal{P}_n^2$ is said to have the same order type.

We say the isomorphism δ is an orientation reversing isomorphism if for any three points the $A, B, C \in \mathcal{P}_n^1$ the orientation $A \longrightarrow B \longrightarrow C \longrightarrow A$ of the triangle ΔABC and the orientation $\delta(A) \longrightarrow \delta(B) \longrightarrow \delta(C) \longrightarrow \delta(A)$ of the triangle $\Delta\delta(A)\delta(B)\delta(C)$ disagree.

DEFINITION 3.10

(Isomorphism between two regions) For $j = 1, 2$, let $\mathcal{P}_n^j = \{P_1^j, \ldots, P_n^j\}$ be two point arrangements in $\mathbb{R}^2$ such that the points form a convex n -gon in the anticlockwise manner $P_1^j \longrightarrow P_2^j \longrightarrow \ldots \longrightarrow P_n^j \longrightarrow P_1^j$ and both of which have generic diagonals. Let R^j be a region in $\mathcal{P}_n^j$, $j = 1, 2$. We say R^1 is isomorphic to R^2 if the line cycle of the regions R^1 and R^2 are same, that is, for any two points $P_{n+1}^j \in R^j$, $j = 1, 2$ the map $\delta : \mathcal{P}_n^1 \cup \{P_{n+1}^1\} \longrightarrow \mathcal{P}_n^2 \cup \{P_{n+1}^2\}$ given by $\delta(P_i^1) = P_i^2$, $1 \le i \le n+1$ is an orientation preserving isomorphism of point arrangements.

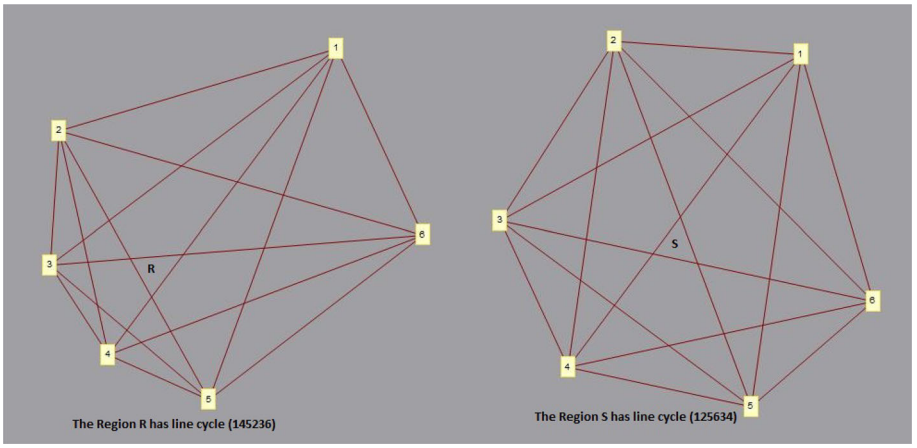

Figure 1. Two non-isomorphic regions R and S in hexagons.

Example 3.11. In Fig. 1, R and S are two non-isomorphic regions of the two hexagons with line cycles (145236), (125634) respectively.

Later in Theorem 4.4 we will observe that if two regions R and S of two convex n-gons respectively are isomorphic then the regions R and S still need not have the same number of sides. The number of sides can vary.

Now we enumerate the 2-standard consecutive cycles.

LEMMA 3.12. *Let $T_n \subset S_n$ be the set of 2-standard consecutive n-cycles in S_n.*

(1) We have

$$\#(T_n) = 2^{n-1} - n.$$

(2) The number of non-isomorphic regions R in a convex n-gon is also $2^{n-1} - n$.

Proof. This proof of (1) follows by counting the cardinality of T_n. If the 2-standard consecutive structure is given by

- $1 < 2 < 3 < \ldots < j$,
- $j+1 < j+2 < \ldots < n$,

then the number of such cycles is given by $\binom{n-1}{j-1} - 1$. Hence the total number is given by

$$\sum_{i=2}^{n-1}\left(\binom{n-1}{i-1} - 1\right) = \sum_{i=0}^{n-1}\left(\binom{n-1}{i} - 1\right) = 2^{n-1} - n.$$

Now we construct for every given 2-standard consecutive cycle, a convex n-gon and a region R inside it which has the given 2-standard consecutive cycle. Let $(1 = a_1 a_2 \ldots a_n)$ be a given 2-standard consecutive cycle. The construction is done as follows. Choose a point P in the plane and draw n lines $L_{a_1}, L_{a_2}, \ldots, L_{a_n}$ passing through P with increasing angles

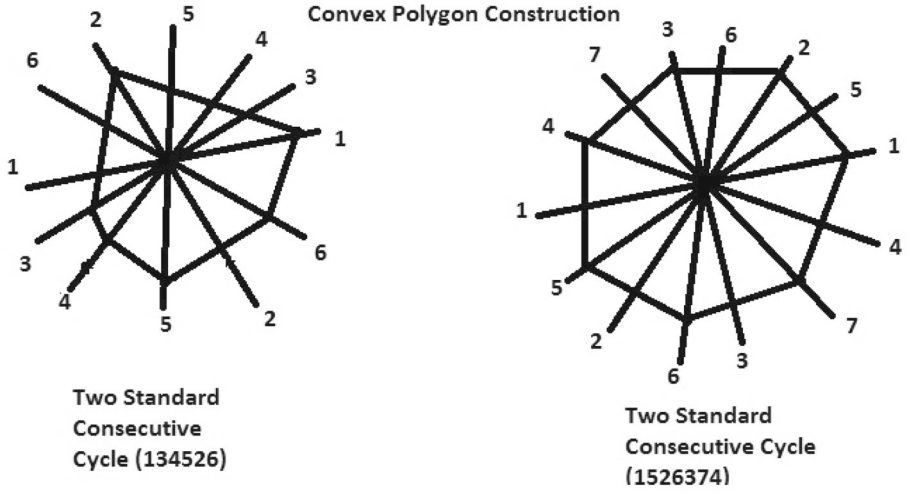

Figure 2. Two examples for $n = 6, 7$ and cycles (134526), (1526374) respectively.

for L_{a_i} with respect to $L_1 = L_{a_1}$ by assuming L_1 is the X-axis. Let $a_{i_j} = j, 1 \leq j \leq n$. Choose a point P_1 on the positive X-axis. Now traverse anticlockwise around the point P cutting the lines $L_{a_2}, L_{a_3} \ldots, L_{a_{(i_2-1)}}$ and choose a point P_2 on the ray that we have reached on $L_{a_{i_2}} = L_2$ and continue this process till we choose a point P_n on a suitable ray of $L_{a_{i_n}} = L_n$. In this process we make sure that the we obtain a convex n-gon $P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P_n \longrightarrow P_1$ in the anticlockwise manner with generic diagonals. We refer to Fig. 2 for two illustrative examples. This proves the lemma. □

4. Definite and Indefinite Cycles and Their Characterization

We have seen in Theorem 2.15 that there are polynomial (in n) given by

$$\frac{(n-1)(n-2)(n^2-3n+12)}{24}$$

number of regions in any convex n-gon with generic diagonals. However in Lemma 3.12 we have seen that there are exponential (in n), precisely $2^{n-1} - n$ number of 2-standard consecutive cycles. Hence we conclude that not every region occurs in every convex n-gon. This leads to definite cycles/regions and indefinite cycles/regions as in Definition 4.1. We see that definite regions are residences of a point $p = n + 1$ such that the matroid D_n is reducible by p and indefinite regions are those such that the matroid D_n is not reducible by p. It happens that, for $n \leq 5$ all regions are definite. This agrees with Proposition 2.10(3) because $n + 1 \leq 6$. From $n = 6$ the indefinite regions start to appear. However for $n \geq 8$ both the types can be combinatorially characterized. Main Theorem **Ω** and Theorem 5.1 in this article characterizes those regions or residences of the point p for which, the matroid D_n is reducible by the point p. Main Theorem **Ω** is stated in Sect. 4 after the required definitions and motivation for these definitions. Theorem 5.1 is stated in the last section.

So we conclude that not every region labeled by 2-standard consecutive cycle occurs in every convex n-gon with generic diagonals. This motivates the following definition.

DEFINITION 4.1

(Definite and Indefinite 2-Standard Consecutive n -cycle) We say a 2-standard consecutive n -cycle $(1 = a_1a_2 \dots a_n)$ is definite if there is a corresponding region R which occurs in every convex n -gon with generic diagonals. Otherwise, we say the cycle is indefinite.

Remark 4.2. In the above definition, we have defined, definite cycles for convex n-gon with generic diagonals. If $\mathcal{C}_n$ is the class of all convex n-gons in the plane and $(\mathcal{C}_n)_{gen}$ is the class of all convex n-gons with generic diagonals then we will see in Theorem 5.1 that the set of definite cycles remains the same for both the classes of convex n-gons. Those which are not definite are the indefinite cycles. The combinatorial descriptions of both types of cycles are given in Theorem **Ω**.

Example 4.3. All regions labelled by 2-standard consecutive cycles of a triangle, quadrilateral and a pentagon are definite. The first occurence of indefinite regions is when $n = 6$. We refer to Fig. 1. The indefinite 2-standard consecutive cycles are given by

$$(145236), (125634).$$

4.1 *Some Properties of* 2*-Standard Consecutive Cycles Associated to Regions*

We state the following theorem.

Theorem 4.4. *Let $\mathcal{P}_n = \{P_1, \dots, P_n\}$ be a point arrangement in $\mathbb{R}^2$ such that the points form a convex n -gon in the anticlockwise manner $P_1 \longrightarrow P_2 \longrightarrow \dots \longrightarrow P_n \longrightarrow P_1$ and which has generic diagonals. Let R be a region with associated* 2*-standard consecutive cycle $(1 = a_1a_2 \dots a_n)$. Let the* 2*-standard consective structure be given by*

$$1 < 2 < \dots < l,$$
$$(l+1) < (l+2) < \dots < n.$$

Then

1. *If P_iP_j is a side of the region R then the elements i, j are consecutive in the cycle $(1 = a_1a_2 \dots a_n)$.*
2. *The* 2*-standard consecutive cycle for the region S on the other side of the region R across P_iP_j is obtained by swapping i, j in the cycle $(1 = a_1a_2 \dots a_n)$.*
3. *Let P_iP_j be the side of the region R and $j \not\equiv i \pm 1 \mod n$. Then either $i \in \{1, 2, \dots, l\}$, $j \in \{l+1, l+2, \dots, n\}$ or $j \in \{1, 2, \dots, l\}$, $i \in \{l+1, l+2, \dots, n\}$.*
4. *The converse is not true, that is, if i, j occur consecutively in $(1 = a_1a_2 \dots a_n)$ such that $j \not\equiv i \pm 1 \mod n$, $1 \le i \le l, l+1 \le j \le n$ then P_iP_j need not be a side of the region R.*
5. *The number of sides of a region R with the same cycle $(1 = a_1a_2 \dots a_n)$ may vary from one convex n -gon to another convex n -gon provided it occurs in them.*
6. *The region R is contained in the triangle $\Delta P_iP_jP_k$ with $i < j < k$ if and only if the cycle $(1 = a_1a_2 \dots a_n)$ contains (ikj) as a sub-cycle.*

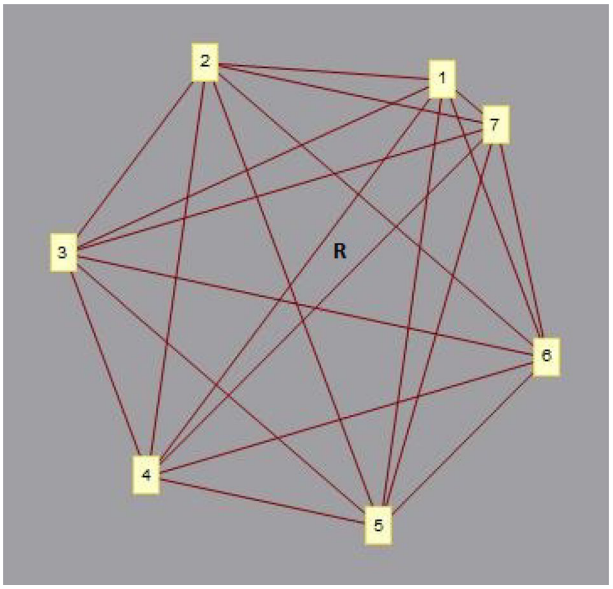

Figure 3. A heptagon and a region R with cycle (1526374) respectively.

Proof. (1) The line cycle does not change for any point in R. Hence choosing a point close to P_iP_j in the region R we conclude that i, j occurs consecutively in the line cycle. Moreover if we give anticlockwise orientation to the sides of R and the directed side is $\overrightarrow{ij}$ then i, j appear next to each other with j first and i second in the the cycle $(1 = a_1a_2 \dots a_n)$.

(2) If we go across the side P_iP_j to a new region S from R then it is clear that there is swap of i, j in the line cycle $(1 = a_1a_2 \dots a_n)$ to obtain the line cycle for S.

(3) If P_iP_j is a side of the region R with $j \not\equiv i \pm 1 \mod n$ then it is not the side of the convex n-gon. So there is a region S adjacent to R across P_iP_j. The cycle of S is obtained by swapping i, j in $(1 = a_1a_2 \dots a_n)$. Now this is 2-standard consecutive if and only if $i \in \{1, 2, \dots, l\}$, $j \in \{l+1, l+2, \dots, n\}$ or vice-versa.

(4) We consider the example in Fig. 3. In this example the region R is a pentagon with cycle (1526374) and the 2-standard consecutive structure $1 < 2 < 3 < 4; 5 < 6 < 7$ with $l = 4$. The numbers $1, 5$ appear consecutively in the cycle of R, however, P_1P_5 is not the side of the region R.

(5) Consider the central region of a regular heptagon. It also has cycle (1526374). This central region is a heptagon where as the region R in Fig. 3 is a pentagon.

(6) Let $Q_1, Q_2, Q_3 \in \mathcal{P}_n$. Let Q_4 be in the interior of the convex n-gon. Then Q_4 is in the interior of the triangle $\Delta Q_1Q_2Q_3$ oriented anticlockwise if and only if the line cycle of Q_4 for the point arrangement $\{Q_1, Q_2, Q_3, Q_4\}$ is (132).

This completes the proof of the theorem. □

4.2 *Characterization of Definite and Indefinite Cycles*

In this section we characterize the definite and indefinite 2-standard consecutive n-cycles combinatorially. We begin with a definition.

DEFINITION 4.5

(The Diagonal Distance of a 2-Standard Consecutive Cycle) Let $(1 = a_1a_2 \dots a_n)$ be a 2-standard consecutive n -cycle. Let the 2-standard consecutive structure be given by

$$1 < 2 < \dots < l,$$
$$l+1 < l+2 < \dots < n.$$

Consider the set S of all pairs $\{i, j\}$ with $1 \leq i \neq j \leq n$ such that

(1) $j \not\equiv i \pm 1 \mod n$,
(2) i, j are consecutive in $(1 = a_1a_2 \dots a_n)$ and
(3) $i \in \{1, \dots, l\}$, $j \in \{l+1, \dots, n\}$ or $j \in \{1, \dots, l\}$, $i \in \{l+1, \dots, n\}$

The diagonal distance of $(1 = a_1a_2 \dots a_n)$ is defined as

$$\min_{\{i,j\} \in S} \{(i-j) \bmod n, (j-i) \bmod n\}$$

Here the residue classes mod n are $\{0, 1, \dots, (n-1)\}$.

Example 4.6. Now we consider an illustration of Theorem 4.7 where the cycles with diagonal distance two are mentioned by taking $n = 6$. There are $2^{6-1} - 6 = 26$ number of 2-standard consecutive cycles. There are 24 cycles with diagonal distance two. Now we list the cycles obtained in Case 1 of Theorem 4.7.

- Move 1 $\longrightarrow$ (134562), (145623), (156234), (162345).
- Move 2 $\longrightarrow$ (132456), (134256), (134526), (134562).
- Move 3 $\longrightarrow$ (124356), (124536), (124563), (132456).
- Move 4 $\longrightarrow$ (123546), (123564), (142356), (124356).
- Move 5 $\longrightarrow$ (123465), (152346), (125346), (123546).
- Move 6 $\longrightarrow$ (162345), (126345), (123645), (123465).

These give $n^2 - 3n = 6^2 - 3.6 = 18$ cycles which are definite. These are the outermost layer regions of the convex n -gon where $n = 6$. Now we list the cycles obtained in Case 2 of Theorem 4.7.

- Move 1 and swap 2, 6 $\longrightarrow$ (134526), (145263), (152634), (126345).
- Move 2 and swap 1, 3 $\longrightarrow$ (124563), (142563), (145263), (145623).
- Move 3 and swap 2, 4 $\longrightarrow$ (142356), (142536), (142563), (134256).
- Move 4 and swap 3, 5 $\longrightarrow$ (125346), (125364), (142536), (124536).
- Move 5 and swap 4, 6 $\longrightarrow$ (123645), (152364), (125364), (123564).
- Move 6 and swap 1, 5 $\longrightarrow$ (156234), (152634), (152364), (152346).

These give in addition $n^2 - 5n = 6^2 - 5.6 = 6$ cycles which are definite. These six cycles correspond to the second outermost layer regions of the convex n -gon where $n = 6$. We also have two indefinite cycles given by (145236), (125634). These total to $18+6+2 = 26$ cycles.

Next we characterize the 2-standard consecutive cycle which has diagonal distance two.

Theorem 4.7. *Let* $(1 = a_1a_2 \ldots a_n)$ *be a* 2*-standard consecutive* n*-cycle with the* 2*-standard consecutive structure given by* $1 < \ldots < l;\ \ l+1 < \ldots < n$*. Then the diagonal distance of the cycle is two if and only if it is obtained from the cycle* $(12 \ldots n)$ *in the following two ways:*

1. *Considering the cyclic notation of the cycle* $(12 \ldots n)$ *on a circle in an anticlockwise manner, and moving* i *forward (anticlockwise) for some* $1 \leq i \leq n-1$ *in a finite number of steps to any position after* $(i+1)$ *and before* $(i-1)$.
2. *From any cycle obtained in the previous step by moving* i*, we swap the adjacent elements* $(i-1), (i+1)$.

Proof. We prove the reverse implication ($\Leftarrow$) first. Any cycle obtained in steps (1), (2) have diagonal distance two since $(i-1), (i+1)$ considered cyclically are consecutive and either $(i-1) \in \{1, \ldots, l\}, i+1 \in \{l+1, l+2, \ldots, n\}$ or $(i+1) \in \{1, \ldots, l\}, i-1 \in \{l+1, l+2, \ldots, n\}$. This proves that the diagonal distance is two.

Now we prove the forward implication. Suppose the diagonal distance is two. Then first we construct a convex n-gon with vertices $P_1 \longrightarrow P_2 \longrightarrow \ldots \longrightarrow P_n \longrightarrow P_1$ in this anticlockwise manner such that the cycle $(1 = a_1a_2 \ldots a_n)$ appears as a region using Fig. 2. Since the diagonal distance is two, there exists $1 \leq i \leq n$ such that $i-1, i+1$ appear consecutively and either $(i-1) \in \{1, \ldots, l\}, i+1 \in \{l+1, l+2, \ldots, n\}$ or $(i+1) \in \{1, \ldots, l\}, i-1 \in \{l+1, l+2, \ldots, n\}$. Now using the convex n-gon, by applying moves of the type mentioned in step (1), (2) on the cycle, we just cross the regions to reach the outside of convex n-gon via crossing either the side $P_{i-1}P_i$ or P_iP_{i+1}. Here the cycle we arrive at must be the cycle $(12 \ldots n)$. Hence this proves the forward implication thereby completing the proof of the theorem. □

Now we prove a lemma which is used in the proof of Theorem **Ω**. This lemma describes certain 2-standard consecutive $(n-1)$-cycles of diagonal distance two which upon adding n gives two standard consecutive n-cycles of diagonal distance two.

LEMMA 4.8. *Let* $7 \leq n \in \mathbb{N}$*. Let* $(1 = a_1a_2 \ldots a_{n-1})$ *be a* 2*-standard consecutive* $(n-1)$*-cycle with diagonal distance two. For some* $1 \leq i \leq n-1$ *let*

$$(i-1) \bmod (n-1), (i+1) \bmod (n-1) \in \{1, \ldots, (n-1)\}$$

appear consecutively in the cycle and suppose $i \notin \{1, 2, (n-2), (n-1)\}$*, that is,*

$$\{(i-1) \bmod (n-1), (i+1) \bmod (n-1)\} \notin \{\{1, n-2\}, \{2, n-1\}, \{1, 3\}, \{n-3, n-1\}\}.$$

If we add n *to the cycle and obtain a* 2*-standard consecutive* n*-cycle then it also has diagonal distance two.*

Proof. If we add n to $(1 = a_1a_2 \ldots a_{n-1})$ in the right of both $(i-1) \bmod (n-1), (i+1) \bmod (n-1)$, the cycle still has diagonal distance two unless $\{(i-1) \bmod (n-1), (i+1) \bmod (n-1)\} \in \{\{2, n-1\}, \{1, n-2\}\}$. We cannot add n after 1 and to the left of both of them as the resulting cycle will not be 2-standard consecutive. Now if we add n in between $(i-1) \bmod (n-1)$ and $(i+1) \bmod (n-1)$ then only the following possibilities occur.

(1) $(1 = a_1a_2 \dots a_{n-1}n) = (1 = a_1 \dots 3n)$ with $a_{n-1} = 3$ or
(2) $(1 = a_1a_2 \dots a_{n-1}n) = (1(n-1)2 \dots (n-2)n)$ with $a_2 = (n-1), a_j = j-1, 3 \leq j \leq n-1$ or
(3) $(1 = a_1a_2 \dots a_{n-1}n) = (12 \dots j(n-1)(j+1) \dots (n-2)n)$ for some $2 \leq j \leq (n-3)$
(4) $(1 = a_1 \dots (n-1)n2 \dots a_{n-1}) = (1j \dots (n-2)(n-1)n2 \dots (j-1))$ with $4 \leq j \leq n-2$ or
(5) $(1 = a_1a_2na_3 \dots a_{n-1}) = (1(n-1)n2 \dots (n-2))$ with $a_2 = (n-1), a_j = j-1, 3 \leq j \leq n-1$ or
(6) $(1 = a_1a_2 \dots a_{n-2}na_{n-1}) = (13 \dots (n-1)n2)$ with $a_{n-1} = 2, a_j = (j+1), 2 \leq j \leq n-2$ or
(7) $(1 = a_1a_2 \dots a_{n-4}a_{n-3}na_{n-2}a_{n-1}) = (123 \dots (n-4)(n-1)n(n-3)(n-2))$ with $a_j = j, 2 \leq j \leq n-4, a_{n-3} = n-1, a_{n-2} = n-3, a_{n-1} = n-2$.
(8) $(1 = a_1a_2 \dots a_{n-3}a_{n-2}na_{n-1}) = (1 \dots j(n-2)(j+1) \dots (n-4)(n-1)n(n-3))$ for some $2 \leq j \leq n-5, a_{n-3} = n-4, a_{n-2} = n-1, a_{n-1} = n-3$.

In these cases we have $\{(i-1) \bmod (n-1), (i+1) \bmod (n-1)\} \in \{\{1, n-2\}, \{2, n-1\}, \{1, 3\}, \{n-1, n-3\}\}$. This proves the lemma. □

Now we state and prove the main theorem of the article. Later in Theorem 5.1, we will formulate the main theorem in the language of matroids.

Theorem Ω. Let $(1 = a_1a_2 \dots a_n)$ be a 2-standard consecutive n-cycle.

(A) The cycle $(1 = a_1a_2 \dots a_n)$ is definite if and only if it is obtained from the cycle $(12 \dots n)$ in the following three ways.

(a) Considering the cyclic notation of the cycle $(12 \dots n)$ on a circle in an anticlockwise manner, and moving i forward (anticlockwise) for some $1 \leq i \leq n-1$ in a finite number of steps to any position after $(i+1)$ and before $(i-1)$. The diagonal distance of these cycles is two.
(b) From any cycle obtained in the previous step by moving i, we swap the adjacent elements $(i-1), (i+1)$. The diagonal distance of these cycles is also two.
(c) $n = 7$ and $(1 = a_1a_2 \dots a_n) = (1526374)$. This is the only cycle which has diagonal distance more than two and is definite. This phenomenon occurs only when $n = 7$.

(B) The cycle $(1 = a_1a_2 \dots a_n)$ is indefinite if and only if there exists

$$1 \leq i_1 < i_2 < i_3 < i_4 < i_5 < i_6 \leq n$$

with the following property that if for some $j \in \{0, 1, 2, 3, 4, 5\}$

$$a_{i_{j+1}} = \min\{a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}, a_{i_5}, a_{i_6}\} \text{ with } a_{i_{j+t}} = a_{i_{((j+t) \bmod 6)}}$$

where $((j+t) \bmod 6) \in \{1, 2, 3, 4, 5, 6\}$ then we have either

$$a_{i_{j+1}} < a_{i_{j+4}} < a_{i_{j+5}} < a_{i_{j+2}} < a_{i_{j+3}} < a_{i_{j+6}}$$

or

$$a_{i_{j+1}} < a_{i_{j+2}} < a_{i_{j+5}} < a_{i_{j+6}} < a_{i_{j+3}} < a_{i_{j+4}}.$$

Here in the subscripts the local cycles (145236), (125634) appear which are the indefinite cycles for $n = 6$.

The proof of Theorem **Ω** is given after the following example.

Example 4.9. We illustrate Theorem **Ω***(B)* in this example. Clearly the cycles (145236), (125634) are indefinite using Theorem **Ω***(B)*. Consider the 2-standard consecutive cycle

$$(1 = a_1a_2a_3a_4a_5a_6a_7a_8) = (15263748).$$

Choose $i_1 = 1, i_2 = 2, i_3 = 4, i_4 = 5, i_5 = 7, i_6 = 8$. Choose $j = 0$. We have

$$a_{i_{j+1}} = a_{i_1} = a_1 = 1 < a_{i_{j+4}} = a_{i_4} = a_5 = 3 < a_{i_{j+5}} = a_{i_5} = a_7 = 4 <$$
$$a_{i_{j+2}} = a_{i_2} = a_2 = 5 < a_{i_{j+3}} = a_{i_3} = a_4 = 6 < a_{i_{j+6}} = a_{i_6} = a_8 = 8.$$

So using Theorem **Ω***(B)* we have that the cycle is indefinite. Also observe that this can be expressed by the fact that (815634) is a sub-cycle.

Consider the 2-standard consecutive cycle

$$(1 = a_1a_2a_3a_4a_5a_6a_7a_8) = (15263784).$$

Choose $i_1 = 2, i_2 = 3, i_3 = 5, i_4 = 6, i_5 = 7, i_6 = 8$. Choose $j = 1$. We have

$$a_{i_{j+1}} = a_{i_2} = a_3 = 2 < a_{i_{j+2}} = a_{i_3} = a_5 = 3 < a_{i_{j+5}} = a_{i_6} = a_8 = 4 <$$
$$a_{i_{j+6}} = a_{i_1} = a_2 = 5 < a_{i_{j+3}} = a_{i_4} = a_6 = 7 < a_{i_{j+4}} = a_{i_5} = a_7 = 8.$$

So using Theorem **Ω***(B)* we have that the cycle is indefinite. Also observe that this can be expressed by the fact that (452378) is a sub-cycle.

Remark 4.10. From Theorem **Ω** and the property given in Theorem **Ω**(B), it is clear that a region R has an indefinite cycle if and only if it does not occur in some convex n-gon with generic diagonals. Also this fact becomes more clear in the proof of Theorem **Ω**.

Proof of Theorem **Ω**. First we prove the reverse ($\Leftarrow$) implication in (A).

The cycles obtained in A(a) and A(b) exactly correspond to the regions for which one of the following n diagonals

$$P_1P_3,\ P_2P_4,\ \ldots,\ P_{n-1}P_1,\ P_nP_2$$

is a side and these clearly are definite regions. These cycles have diagonal distance exactly two. There are $2n^2 - 8n$ such cycles with corresponding regions. For $n = 7$ there are $2^{7-1} - 7 = 57$ number of 2-standard consecutive cycles. There are $\frac{(7-1)(7-2)(7^2-3*7+12)}{24} = 50$ regions in a heptagon with generic diagonals. Now in Fig. 1 we have seen for the hexagon that the cycles (145236) and (125634) are indefinite and mutually exclusive, that is, if one occurs then the other does not occur. Extending this scenario for $n = 7$ we conclude that there are seven pairs of mutually exclusive 2-standard consecutive cycles. They are obtained by cyclically shifting as follows.

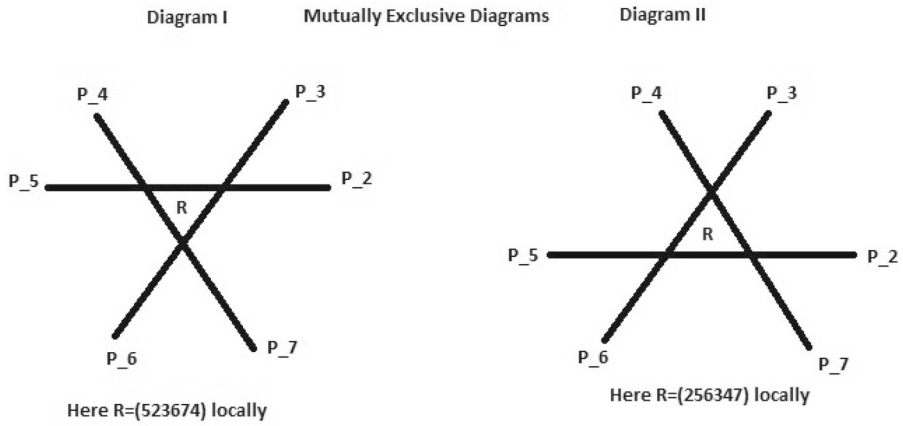

Figure 4. Local Triangles ignoring 1 containing mutually exclusive cycles/regions (1523674), (1256347) respectively.

(I) First pair (1523674), (1256347). We ignore 1 and the oriented triangle $\Delta\overrightarrow{P_5P_2}\overrightarrow{P_3P_6}\overrightarrow{P_7P_4}$ containing the region (1523674) is oriented clockwise, where as, the oriented triangle $\Delta\overrightarrow{P_5P_2}\overrightarrow{P_3P_6}\overrightarrow{P_7P_4}$ containing the region (1256347) is anticlockwise. We illustrate this in Fig. 4.

(II) Second pair (1526347), (1236745). We ignore 2 and the oriented triangle $\Delta\overrightarrow{P_1P_5}\overrightarrow{P_6P_3}\overrightarrow{P_4P_7}$ containing the region (1526347) is oriented clockwise, where as, the oriented triangle $\Delta\overrightarrow{P_1P_5}\overrightarrow{P_6P_3}\overrightarrow{P_4P_7}$ containing the region (1236745) is anticlockwise.

(III) Third pair (1263745), (1562347). We ignore 3 and the oriented triangle $\Delta\overrightarrow{P_5P_1}\overrightarrow{P_2P_6}\overrightarrow{P_7P_4}$ containing the region (1263745) is oriented clockwise, where as, the oriented triangle $\Delta\overrightarrow{P_5P_1}\overrightarrow{P_2P_6}\overrightarrow{P_7P_4}$ containing the region (1562347) is anticlockwise.

(IV) Fourth pair (1562374), (1267345). We ignore 4 and the oriented triangle $\Delta\overrightarrow{P_1P_5}\overrightarrow{P_6P_2}\overrightarrow{P_3P_7}$ containing the region (1562374) is oriented clockwise, where as, the oriented triangle $\Delta\overrightarrow{P_1P_5}\overrightarrow{P_6P_2}\overrightarrow{P_3P_7}$ containing the region (1267345) is anticlockwise.

(V) Fifth pair (1526734), (1456237). We ignore 5 and the oriented triangle $\Delta\overrightarrow{P_4P_1}\overrightarrow{P_2P_6}\overrightarrow{P_7P_3}$ containing the region (1526734) is oriented clockwise, where as, the oriented triangle $\Delta\overrightarrow{P_4P_1}\overrightarrow{P_2P_6}\overrightarrow{P_7P_3}$ containing the region (1456237) is anticlockwise.

(VI) Sixth pair (1452637), (1256734). We ignore 6 and the oriented triangle $\Delta\overrightarrow{P_1P_4}\overrightarrow{P_5P_2}\overrightarrow{P_3P_7}$ containing the region (1452637) is oriented clockwise, where as, the oriented triangle $\Delta\overrightarrow{P_1P_4}\overrightarrow{P_5P_2}\overrightarrow{P_3P_7}$ containing the region (1256734) is anticlockwise.

(VII) Seventh pair (1256374), (1452367). We ignore 7 and the oriented triangle $\Delta\overrightarrow{P_4P_1}\overrightarrow{P_2P_5}\overrightarrow{P_6P_3}$ containing the region (1256374) is oriented clockwise, where as, the oriented triangle $\Delta\overrightarrow{P_4P_1}\overrightarrow{P_2P_5}\overrightarrow{P_6P_3}$ containing the region (1452367) is anticlockwise.

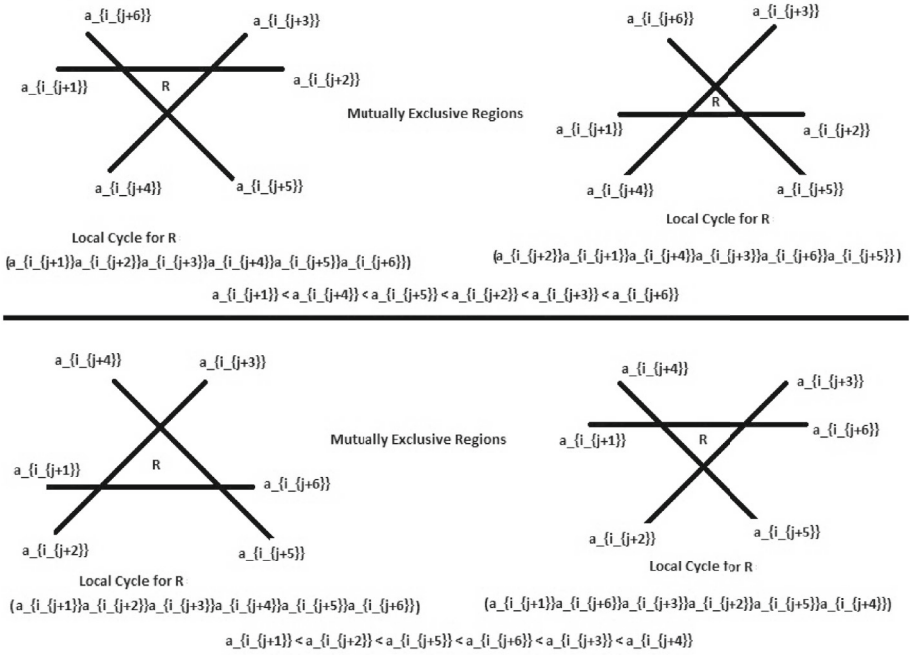

Figure 5. Mutually exclusive indefinite regions.

Now there are $2n^2 - 8n = 2*7^2 - 8*7 = 42$ definite cycles whose regions definitely occur using A(a),A(b). These cycles can be written down. Out of the remaining 15 cycles there are seven mutually exclusive pairs. Totally there are 50 regions that occur in a heptagon with generic diagonals. Hence there is one more definite cycle which occurs that is cycle (1526374). This completes the proof of reverse implication ($\Leftarrow$) of Theorem **Ω**(A).

Now we prove the reverse implication ($\Leftarrow$) in (B). In both the cases mentioned in (B) we can obtain a mutually exclusive cycle for the cycle $(1 = a_1 a_2 \dots a_n)$ by orienting the local triangle the other way similar to the diagrams given in Fig. 4. We illustrate this in Fig. 5.

This proves that the cycle $(1 = a_1 a_2 \dots a_n)$ is indefinite. This completes the proof of reverse implication ($\Leftarrow$) of Theorem **Ω**(B).

To complete the proof of Theorem **Ω** it is enough to prove that if for a 2-standard consecutive cycle $(1 = a_1 a_2 \dots a_n)$, the diagonal distance is more than two and $n \geq 8$ (Note: $n \neq 7$) then the cycle is indefinite and satisfies the property mentioned in Theorem **Ω**(B).

First we show that for $n = 8$ the only definite cycles are those with diagonal distance two explicitly. For this we list all the 120 number of 2-standard consecutive cycles as a union of fifteen orbits each containing eight cycles under the cyclic shift action of $\mathbb{Z}/8\mathbb{Z}$. The cycles in the first eight orbits $1, \dots, 8$ have diagonal distance two and satisfy the conditions of Theorem **Ω** in A(a),A(b). The cycles in the remaining seven orbits $9, \dots, 15$ satisfy the conditions of Theorem **Ω**(B). The cycles in these orbits do not have diagonal distance two. We consider the one standard consecutive cycle (12345678) on a circle in anticlockwise cyclic manner.

(1) Move element i forward (anticlockwise) by one position in (12345678) for $1 \leq i \leq 8$. This is also same as moving another element i forward by six positions in (12345678) for $1 \leq i \leq 8$. (13456782), (13245678), (12435678), (12354678), (12346578), (12345768), (12345687), (18234567).

(2) Move element i forward by two positions in (12345678) for $1 \leq i \leq 8$. This is also same as moving another element i forward by one position in (12345678) and swapping $(i-1)$, $(i+1)$ considered cyclically for $1 \leq i \leq 8$. They are given as: (14567823), (13425678), (12453678), (12356478), (12346758), (12345786), (17234568), (12834567).

(3) Move element i forward by three positions in (12345678) for $1 \leq i \leq 8$. (15678234), (13452678), (12456378), (12356748), (12346785), (16234578), (12734568), (12384567).

(4) Move element i forward by four positions in (12345678) for $1 \leq i \leq 8$. (16782345), (13456278), (12456738), (12356784), (15234678), (12634578), (12374568), (12348567).

(5) Move element i forward by five positions in (12345678) for $1 \leq i \leq 8$. (17823456), (13456728), (12456783), (14235678), (12534678), (12364578), (12347568), (12345867).

(6) Move element i forward by two positions in (12345678) and swap $(i-1)$, $(i+1)$ considered cyclically for $1 \leq i \leq 8$. This is also same as moving another element i forward by five positions in (12345678) and swapping $(i-1)$, $(i+1)$ considered cyclically for $1 \leq i \leq 8$. They are given as: (14567283), (14256783), (14253678), (12536478), (12364758), (12347586), (17234586), (17283456).

(7) Move element i forward by three positions in (12345678) and swap $(i-1)$, $(i+1)$ considered cyclically for $1 \leq i \leq 8$. (15672834), (14526783), (14256378), (12536748), (12364785), (16234758), (12734586), (17238456).

(8) Move element i forward by four positions in (12345678) and swap $(i-1)$, $(i+1)$ considered cyclically for $1 \leq i \leq 8$. (16728345), (14562783), (14256738), (12536784), (15236478), (12634758), (12374586), (17234856).

(9) The following cycles satisfy the condition of Theorem **Ω**(B). For example the cycle (12563478) contains the sub-cycle (125634). (12563478), (12367458), (12347856), (16723458), (12783456), (14567238), (12567834), (14523678).

(10) The following cycles satisfy the condition of Theorem **Ω**(B). For example the cycle (12563748) contains the sub-cycle (125634). (12563748), (12367485), (16234785), (16273458), (12738456), (15672384), (15267834), (14526378).

(11) The following cycles satisfy the condition of Theorem **Ω**(B). For example the cycle (12567348) contains the sub-cycle (125634). (12567348), (12367845), (15623478), (12673458), (12378456), (15672348), (12678345), (14562378).

(12) The following cycles satisfy the condition of Theorem **Ω**(B). For example the cycle (12567384) contains the sub-cycle (125634). (12567384), (15236784), (15263478), (12637458), (12374856), (16723485), (16278345), (14562738).

(13) The following cycles satisfy the condition of Theorem **Ω**(B). For example the cycle (12563784) contains the sub-cycle (125634).
(12563784), (15236748), (12634785), (16237458), (12734856), (16723845), (15627834), (14526738).

(14) The following cycles satisfy the condition of Theorem **Ω**(B). For example the cycle (12637485) contains the sub-cycle (126745) locally ignoring 3, 8.
(12637485), (16237485), (16273485), (16273845), (15627384), (15267384), (15263784), (15263748).

(15) The following cycles satisfy the condition of Theorem **Ω**(B). For example the cycle (12673845) contains the sub-cycle (126745) locally ignoring 3, 8.
(12673845), (15623784), (15267348), (12637845), (15623748), (12673485), (16237845), (15627348).

This completes the proof of Theorem **Ω** for $n = 8$ the base case of the induction step.

Now we show by induction on $k = n \geq 9$ that if the cycle is indefinite then the diagonal distance is not two and satisfies the criterion given in Theorem **Ω**(B).

Using Lemma 4.8, we first consider cycles of diagonal distance two and hence definite cycles for $k = n-1 \geq 8$ of the form $(1 = a_1 a_2 \ldots a_{n-1})$ which satisfy one of the following properties.

(a) The cycle contains 2 and $(n-1)$ consecutively with $(n-1)$ first and 2 next (has diagonal distance two).
(b) The cycle contains 2 and $(n-1)$ consecutively with 2 first and $(n-1)$ next (has diagonal distance two).
(c) The cycle contains 1 and $(n-2)$ consecutively with 1 first and $(n-2)$ next to it (which has diagonal distance two).
(d) The cycle contains 1 and $(n-2)$ consecutively with 1 first and $(n-2)$ at the end (which has diagonal distance two).
(e) The cycle contains $(n-1)$ and $(n-3)$ consecutively with $(n-1)$ first and $(n-3)$ next (has diagonal distance two).
(f) The cycle contains $(n-3)$ and $(n-1)$ consecutively with $(n-3)$ first and $(n-1)$ next (has diagonal distance two).
(g) The cycle contains 1 and 3 consecutively with 1 first and 3 next to it (which has diagonal distance two).
(h) The cycle contains 1 and 3 consecutively with 1 first and 3 at the end (which has diagonal distance two).

These cycles upon adding n may or may not remain definite. The remaining definite cycles upon adding n will remain definite using Lemma 4.8. We will later consider indefinite $(n-1)$-cycles. We mention the various cases (i)-(xvi) and prove in each case that upon adding n the resulting 2-standard consecutive cycle will either have diagonal distance two and hence remain definite or it does not have diagonal distance two and satisfies the criterion of Theorem **Ω** (so becomes indefinite).

(i) If the cycle $(1 = a_1 a_2 \ldots a_{n-1})$ contains 2, $(n-1)$ consecutively with $(n-1)$ first and 2 next (has diagonal distance two) then the cycle is of the following form

$$(1j \ldots (n-2)(n-1)23 \ldots (j-1)) \text{ for some } 4 \leq j \leq n-2 \text{ or}$$

$$\text{either } (134 \ldots (n-1)2) \text{ or } (1(n-1)23 \ldots (n-2)).$$

It is clear that upon adding n to $(134\ldots(n-1)2)$ the resulting 2-standard consecutive cycle has diagonal distance two and hence definite. If we add n in between $(n-1)$ and 2 or just next to 2 in any of these the resulting 2-standard consecutive cycle also has diagonal distance two and hence definite. If we add n to $(1(n-1)23\ldots(n-2))$ anywhere after 3 then the resulting 2-standard consecutive cycle also has diagonal distance two and hence definite. If we add n to $(1j\ldots(n-2)(n-1)23\ldots(j-1))$ with $4 \leq j \leq n-2$, anywhere after 3 then the resulting 2-standard consecutive cycle does not have diagonal distance two and is indefinite because it has $(23n1(n-2)(n-1))$ as a sub-cycle which satisfies the criterion of Theorem **Ω**(B).

(ii) 2, $(n-1)$ appear in this order next to 1 as follows. Now consider the cycle

$$\big(12(n-1)34\ldots(n-4)(n-3)(n-2)\big).$$

If we add n anywhere after $(n-1)$ and before $(n-3)$ the resulting 2-standard cycle does not have diagonal distance two and is indefinite because it has $(12(n-1)n(n-3)(n-2))$ as a sub-cycle which satisfies the criterion of Theorem **Ω**(B). If we add n in between $(n-3)$ and $(n-2)$ or after $(n-2)$ then the cycle is definite and has diagonal distance two.

(iii) 2, $(n-1)$ appear in this order in the middle as follows. Now consider for some $3 \leq a \leq n-4$ the cycle

$$(1(a+2)\ldots(n-2)2(n-1)3\ldots a(a+1)).$$

If we add n anywhere between $(n-1)$ and a, then the resulting 2-standard cycle does not have diagonal distance two and is indefinite because it has $(12(n-1)na(a+1))$ as a sub-cycle which satisfies the criterion of Theorem **Ω**(B). If we add n in between a and $a+1$ and obtain the 2-standard consecutive cycle $(15\ldots(n-2)2(n-1)3n4)$ for $a=3$ then it does not have diagonal distance two and is indefinite because it has $(23n156)$ as a sub-cycle which satisfies the criterion of Theorem **Ω**(B) since $n \geq 8$. This argument does not work for $n=7$. This is because (152634) has diagonal distance two for $k=6$ and if we add $n=7$ in between $(n-1)=6$ and 4 we get (1526374) which does not have (237156) as a sub-cycle. In fact, (1526374) does not have diagonal distance two and also does not satisfy the criterion of Theorem **Ω**(B). It is a definite cycle for $n=7$ and this is the only exception phenomenon. If we add n after $(a+1)$ or for $a>3$ we add n in between a and $(a+1)$ then the resulting 2-standard cycle does not have diagonal distance two and is indefinite because it has $(34n1(n-2)(n-1))$ as a sub-cycle which satisfies the criterion of Theorem **Ω**(B).

(iv) 2, $(n-1)$ appear in this order at the last but one or at the end positions as follows. Now consider the cycles

$$(14\ldots(n-2)2(n-1)3) \text{ and } (134\ldots(n-2)2(n-1)).$$

These cycles are definite as 1, 3 appear consecutively and has diagonal distance two.

(v) 1, $(n-2)$ appear in the beginning and $(n-1)$ appears at the end as follows. Now consider the cycle

$$\big(1(n-2)23\ldots(n-3)(n-1)\big).$$

n can only be added at the end and the resulting 2-standard cycle is definite and has diagonal distance two and it has $(n-3)$, $(n-1)$ as consecutive.

(vi) 1, $(n-2)$ appear in the beginning and $(n-1)$ appears at the last but one position as follows. Now consider the cycle

$$\big(1(n-2)23\ldots(n-4)(n-1)(n-3)\big).$$

If n is added at the end then the the resulting 2-standard cycle is definite and has diagonal distance two and it has $(n-3)$, $(n-1)$ as consecutive. If n is added in between $(n-1)$ and $(n-3)$ then the resulting 2-standard cycle does not have diagonal distance two and is indefinite because it has $(23(n-1)n(n-3)(n-2))$ as a sub-cycle which satisfies the criterion of Theorem $\mathbf{\Omega}$(B).

(vii) 1, $(n-2)$ appear in the beginning and $(n-1)$ appears in between 3 and $(n-4)$ as follows. Now consider the cycle

$$\big(1(n-2)23\ldots(n-1)\ldots(n-4)(n-3)\big).$$

If we add n at the end then the resulting 2-standard cycle does not have diagonal distance two and is indefinite because it has $((n-4)(n-3)n1(n-2)(n-1))$ as a sub-cycle which satisfies the criterion of Theorem $\mathbf{\Omega}$(B). If we add n after $(n-1)$ and before $(n-3)$ then the resulting 2-standard cycle does not have diagonal distance two and is indefinite because it has $(23(n-1)n(n-3)(n-2))$ as a sub-cycle which satisfies the criterion of Theorem $\mathbf{\Omega}$(B).

(viii) 1, $(n-2)$ appear in the beginning and $(n-1)$ appears in between 2 and 3 as follows. Now consider the cycle

$$\big(1(n-2)2(n-1)34\ldots(n-4)(n-3)\big).$$

If we add n anywhere after 4 then the resulting 2-standard cycle does not have diagonal distance two and is indefinite because it has $(34n1(n-2)(n-1))$ as a sub-cycle which satisfies the criterion of Theorem $\mathbf{\Omega}$(B). If we add n in between $(n-1)$ and before 4 then the resulting 2-standard cycle does not have diagonal distance two and is indefinite because it has $(12(n-1)n45)$ (Note: $n \geq 8$) as a sub-cycle which satisfies the criterion of Theorem $\mathbf{\Omega}$(B). For $n = 7$, this argument does not work. This is because (152634) has diagonal distance two for $k = 6$ and if we add $n = 7$ in between $(n-1) = 6$ and 4 we get (1526374) which does not have (126745) as a sub-cycle. In fact (1526374) does not have diagonal distance two and also does not satisfy the criterion of Theorem $\mathbf{\Omega}$(B). It is a definite cycle for $n = 7$ and this is the only exception phenomenon.

(ix) 1, $(n-2)$ appear in the beginning and $(n-1)$ appears in between $(n-2)$ and 2 as follows. Now consider the cycle

$$\big(1(n-2)(n-1)23\ldots(n-4)(n-3)\big).$$

If we add n anywhere after 3 then the resulting 2-standard cycle does not have diagonal distance two and is indefinite because it has $(23n1(n-2)(n-1))$ as a sub-cycle which satisfies the criterion of Theorem $\mathbf{\Omega}$(B). If we add n in between 2

and 3 or in between $(n-1)$ and 2 then the the resulting 2-standard cycle has diagonal distance two and is definite with 2, n as consecutive.

(x) 1, $(n-2)$ appear consecutively with 1 first and $(n-2)$ at the end as follows. Now consider the cycle

$$\big(1\ldots(n-1)\ldots(n-2)\big).$$

If we add n at the end after $(n-2)$ or just before $(n-2)$ then the resulting two standard cycle has diagonal distance two and is definite. If we add n after $(n-1)$ and before $(n-2)$ then the cycles $(1(n-1)2\ldots n\ldots(n-3)(n-2))$, $(1(n-1)n2\ldots(n-3)(n-2))$ has diagonal distance two and are definite. The cycle $(12\ldots(n-1)\ldots n\ldots(n-3)(n-2))$ does not have diagonal distance two and is indefinite because it has $(12(n-1)n(n-3)(n-2))$ as a sub-cycle which satisfies the criterion of Theorem **Ω**(B).

(xi) $(n-1)$, $(n-3)$ appear consecutively in this order and $(n-2)$ appears after $(n-3)$. Now consider the cycle

$$\big(123\ldots(n-1)(n-3)(n-2)\big).$$

If we add n after $(n-3)$ then the resulting 2-standard cycle has diagonal distance two and is definite. If we add n in between $(n-1)$ and $(n-3)$ then the resulting 2-standard cycle does not have diagonal distance two and is indefinite because it has $(12(n-1)n(n-3)(n-2))$ as a sub-cycle which satisfies the criterion of Theorem **Ω**(B).

(xii) $(n-1)$, $(n-3)$ appear consecutively in this order and $(n-2)$ appears before $(n-3)$. Now consider the cycle

$$\big(1\ldots(n-2)\ldots(n-1)(n-3)\big).$$

If we add n after $(n-3)$ then the resulting 2-standard cycle has diagonal distance two and is definite. If we add n in between $(n-1)$ and $(n-3)$ then the cycles $\big(1\ldots(n-5)(n-4)(n-2)(n-1)n(n-3)\big)$ and $\big(1\ldots(n-5)(n-2)(n-4)(n-1)n(n-3)\big)$ has diagonal distance two and are definite. The cycle $\big(1\ldots(n-2)\ldots(n-5)(n-4)(n-1)n(n-3)\big)$ does not have diagonal distance two and is indefinite because it has $((n-5)(n-4)(n-1)n(n-3)(n-2))$ as a sub-cycle which satisfies the criterion of Theorem **Ω**(B).

(xiii) $(n-3)$, $(n-1)$ appear consecutively in this order and $(n-2)$ appears before $(n-3)$. Now consider the cycle

$$\big(1\ldots(n-2)\ldots(n-3)(n-1)\big).$$

Now n has to be added at the end and the resulting 2-standard cycle has diagonal distance two and is definite.

(xiv) $(n-3)$, $(n-1)$ appear consecutively in this order and $(n-2)$ appears after $(n-3)$. Now consider the cycle

$$\big(1\ldots(n-1)(n-3)(n-2)\big).$$

If n is added after $(n-3)$ then the resulting 2-standard cycle has diagonal distance two and is definite. If n is added in between $(n-1)$ and $n-3$ then the resulting 2-standard

cycle does not have diagonal distance two and is indefinite because it has $(12(n-1)n(n-3)(n-2))$ as a sub-cycle which satisfies the criterion of Theorem **Ω**(B).

(xv) 1, 3 appear consecutive with 1 first and 3 just next to it. Now consider the cycle

$$\big(13\ldots2\ldots(n-1)\big),\big(13\ldots(n-1)2\big).$$

Now n must be added at the end in $\big(13\ldots2\ldots(n-1)\big)$ and the resulting 2-standard cycle has diagonal distance two and is definite. In $\big(13\ldots(n-1)2\big)$, n must be added after $(n-1)$ and the resulting 2-standard cycle has diagonal distance two and is definite.

(xvi) 1, 3 appear consecutive with 1 first and 3 at the end. Now consider the cycle

$$\big(1\ldots2\ldots(n-1)3\big).$$

If n is added in between $(n-1)$ and 3 then the resulting 2-standard cycle has diagonal distance two and is definite. If n is added after 3 at the end then the cycles $(1425\ldots(n-1)3n)$, $(1245\ldots(n-1)3n)$ have diagonal distance two and are definite. The cycle $(145\ldots2\ldots(n-1)3n)$ does not have diagonal distance two and is indefinite because it has $(23n145)$ as a sub-cycle which satisfies the criterion of Theorem **Ω**(B).

If a 2-standard consecutive $(n-1)$-cycle $(1=a_1a_2\ldots a_{n-1})$ is already indefinite satisfying the criterion of Theorem **Ω**(B) then it satisfies the criterion of Theorem **Ω**(B) after adding n and obtain a 2-standard consecutive n-cycle. We will now show that it does not have diagonal distance two. Let $(1=a_1\ldots a_{n-1})$ be an indefinite $(n-1)$-cycle. By induction it does not have diagonal distance two. After adding n we show that n cannot be adjacent to 2 or $(n-2)$. Suppose the $(n-1)$-cycle is given by $(1=a_1\ldots(n-2)\ldots(n-1)\ldots a_{n-1})$. Then n should appear after $(n-1)$ unless the cycle is $(12\ldots(n-3)n(n-2)(n-1))$ or $(12\ldots(n-2)n(n-1))$, which is impossible. Hence n and $(n-2)$ cannot be adjacent. Suppose $(n-2)$ appears after $(n-1)$ in the indefinite $(n-1)$-cycle $(1=a_1\ldots a_{n-1})$. Then it is given by $(1=a_1\ldots(n-1)\ldots(n-2)=a_{n-1})$, which has diagonal distance two. Hence a contradiction. By a similar reasoning n cannot appear adjacent to 2. So the resulting 2-standard consecutive n-cycle does not have diagonal distance 2. This completes the proof of main Theorem **Ω**. □

5. The Main Theorem in Terms of Matroids

We state and prove the theorem.

Theorem 5.1. *Let $n \geq 3$ be a positive integer.*

*(1) The map $(\pi_p)_{|_{\mathcal{R}_{gen}(D_n)}} : \mathcal{R}_{gen}(D_n) \longrightarrow \mathcal{R}_{gen}(C_n = D_n \backslash p)$ induced by deletion of the point p is surjective if and only if the 2-standard consecutive cycle associated to the region/residence of $p = n+1$ is a definite cycle as given in Theorem **Ω**(A).*

*(2) The map $\pi_p : \mathcal{R}(D_n) \longrightarrow \mathcal{R}(C_n = D_n \backslash p)$ is surjective if and only if the 2-standard consecutive cycle associated to the region/residence of $p = n+1$ is a definite cycle as given in Theorem **Ω**(A).*

Proof. We prove (1). Clearly the region/cycle R is definite as given in Theorem **Ω**(A) if and only if it occurs in every convex n-gon with generic diagonals. Hence p is in such a region R if and only if the map $(\pi_p)_{|_{\mathcal{R}_{gen}(D_n)}} : \mathcal{R}_{gen}(D_n) \longrightarrow \mathcal{R}_{gen}(C_n = D_n \backslash p)$ is surjective.

We prove (2). Suppose $\pi_p : \mathcal{R}(D_n) \longrightarrow \mathcal{R}(C_n)$ is surjective. Then $\mathcal{R}_{gen}(D_n) \subseteq \pi_p^{-1}(\mathcal{R}_{gen}(C_n)) \subseteq \mathcal{R}(D_n)$ and $\pi_p : \pi_p^{-1}(\mathcal{R}_{gen}(C_n)) \longrightarrow \mathcal{R}_{gen}(C_n)$ is surjective. Since the fibre is associated to a convex open set which is the residence of a point p, we can also choose p in a further general position in the same convex open set. So we obtain that actually the following map

$$(\pi_p)_{|_{\mathcal{R}_{gen}(D_n)}} : \mathcal{R}_{gen}(D_n) \longrightarrow \mathcal{R}_{gen}(C_n = D_n \backslash p)$$

is surjective. Now we use (1) to conclude that R is a definite region and its associated cycle is a definite cycle as described in Theorem **Ω**(A).

Conversely the regions corresponding to definite cycles as described in Theorem **Ω**(A) appear in the class $\mathcal{C}_n$ of all convex n-gons, not just the class $(\mathcal{C}_n)_{gen}$ of convex n-gons with generic diagonals. Note that for $n = 7$ the region corresponding to the definite cycle (1526374) also occurs in the class of all convex 7-gons. This is because, in any convex heptagon $P_1 \to P_2 \to P_3 \to P_4 \to P_5 \to P_6 \to P_7 \to P_1$ with no three vertices P_i, P_j, P_k with $1 \leq i < j < k \leq 7$ are collinear, it is easy to see that the interior of any three triangles of the form $\Delta P_i P_{i+3} P_{i+4}$, $\Delta P_j P_{j+3} P_{j+4}$ and $\Delta P_k P_{k+3} P_{k+4}$ has a nonempty intersection. (Note here for the vertices of these triangles, the subscripts are considered modulo 7 in the set $\{1, 2, 3, 4, 5, 6, 7\}$). Hence by using page 102, Helly's theorem [16], all the seven triangles $\Delta P_i P_{i+3} P_{i+4}$ for $1 \leq i \leq 7$ intersect with nonempty interior. So the cycle (1526374) is a definite cycle and occurs in the class of all convex 7-gons. Hence (2) follows. □

References

[1] Freeman J W, The number of regions determined by a convex polygon, *Math. Magazine*, **49(1)**, (1976), pp. 23–25,

[2] Honsberger R, Mathematical Gems I, The Dolciani Mathematical Expositions, *Math. Assoc. Am.* (1973), xi+176pp, ISBN-10: 0883853019, MR0419117

[3] Herman J, Kucera R and Simsa J, Counting and configurations: problems in combinatorics, arithmetic, and geometry, CMS Books in Mathematics (2003) 12, Springer-Verlag New York, xii+392pp, ISBN-13: 978-0-387-95552-0, https://doi.org/10.1007/978-1-4757-3925-1, MR1950450

[4] *A006522, The on-line encyclopedia of integer sequences*, OEIS Foundation Inc. (2019), http://oeis.org/A006522

[5] Brass P, Moser W O J and Pach J, Problems on points in general position, in: Research Problems in Discrete Geometry (2005) Springer New York, NY, xii+499pp, ISBN-13: 978-0-387-23815-9 https://doi.org/10.1007/0-387-29929-7_9, MR2163782

[6] Ziegler G M, Oriented matroids today, *Electron. J. Combinatorics*, **3**(1) 1996

[7] Crapo H H, Single-element extensions of matroids, *J. Res. Natl. Bur. Standards. Sect. B. Mathematics Mathematical Phys.***69B**(1) and (2) (1965) https://nvlpubs.nist.gov/nistpubs/jres/69B/jresv69Bn1-2p55_A1b.pdf, MR0190045

[8] Björner A, Vergnas M L, Sturmfels B, White N and Ziegler G M, Oriented Matroids, Encylopedia of Mathematics and Applications, **46**, Second Edition, Cambridge University Press, 1999, xii+548pp, ISBN: 0-521-77750-X, https://doi.org/10.1017/CBO9780511586507, MR1744046

[9] Kelly L M and Moser W O J, On the number of ordinary lines determined by *n points*, *Canadian J. Math.***10** (1958) 210-219
[10] Goodman J E and Pollack R, On the combinatorial classification of non-degenerate configurations in the plane, *J. Combinatorial Theory Series A*, **29**(2) (1980) 220-235
[11] Goodman J E and Pollack R, Semispaces of configurations, cell complexes of arrangements, *J. Combinatorial Theory Series A*, *37*(3) (1984) 257-293
[12] Aigner M, and Ziegler G M, Proofs from THE BOOK (2010) Springer (India) Private Limited, ISBN-13: 978-81-8489-533-9 https://doi.org/10.1007/978-3-662-44205-0, MR1723092
[13] Vergnas M L, Convexity in oriented matroids, *J. Combinatorial Theory, Series B*, **29**(2) (1980) 231-243
[14] Richter J and Sturmfels B, On the topology and geometric construction of oriented matroids and convex polytopes, *Trans. Am. Math. Soc.***325**(1) (1991) 389-412
[15] Welsh D J A, On the hyperplanes of a matroid, *Proc. Cambridge Philosoph. Soc.***65** (1969) 11-18
[16] Danzer L, Grünbaum B and Klee V, Helly's theorem and its relatives, Proceedings of Symposia in Pure Mathematics (1963) Vol. VII, Convexity, pp. 101-180, American Mathematical Society, Providence RI, https://www.ams.org/books/pspum/007/0157289, MR0157289

Note from the Chief Editor: The work is done when the author is a Post Doctoral Fellow at HRI, Allahabad

COMMUNICATING EDITOR: C S Rajan

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.